1.2.26 Show that G is bipartite iff $\forall H \subset G$ $\exists$ an indep set of size at least half of $V(H)$. (An inductive proof!)

Pf: If G bipartite, let $\{X, Y\}$ bipart. Given H, let

$$V(H) = V(H) \cap X \cup V(H) \cap Y$$

$$|V(H)| = |V(H) \cap X| + |V(H) \cap Y| \qquad \star 1$$

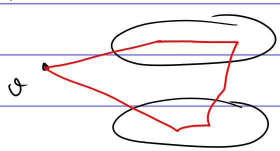
$\Rightarrow$ at least one of them must satisfy $|V(H) \cap X| \geq \frac{V(H)}{2}$

otherwise $\star 1$ cannot hold

$\Leftarrow$ By induction on $n(G)$. Assume the result for $n(G) = 1, 2, \ldots k-1$

ie if $\forall H \subset G$ $\exists$ an indep set in H of size at least half of $V(H)$ then G is bipartite.

Let $n(G) = k$, Pick any vertex v, and consider $G - v$. every subgr $H \subset G - v \subset G$, so it satisfies the condition and thus $G - v$ is bipartite



Suppose G is not bipartite then $\exists$ an odd cycle containing v (shown in red)

Let G' be this odd cycle C, labeled $\{v_1, \ldots, v_{2k+1}\}$.

Let $X = \{v_1, v_3, \ldots, v_{2k-1}, v_{2k+1}\}$ $\quad |X| = k+1 \quad |Y| = k$

$Y = \{v_2, \ldots, v_{2k}\}$

Let I be an indep set in C. Each $v \in Y$ has two neighbors in X

Consider $I \cap Y = \{v_{i_1} \cdots v_{i_j}\}$ where $i_1 < i_2 \cdots < i_j$

$\{v_{i_1}, v_{i_j}\}$ has at least 3 neighbours and in general it should be at least $j+1$ neighbours. This can be proved by induction: if at some point r, the neighbours of v_{i_r} are already in $\{v_{i_1}, \dots v_{i_{r-1}}\}$ then v_{i_r-2} and v_{i_r+2} must already be in the set, which is a contradiction.

So if $|I \cap Y| = j$ then $|I \cap X| \leq |X| - (j+1) = k-j$

$\Rightarrow$ $|I \cap X| + |I \cap Y| \leq k$. However $\frac{|C|}{2} = \frac{2k+1}{2} > k$